**Sistemas Digitais I**

LESI - 2º ano

Lesson 3 - Boolean Algebra

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# 3. Boolean Algebra
*- Introduction -*

- The success of computer technology is primarily based on simplicity of designing digital circuits and ease of their manufacture.
- Digital circuits are composed of basic processing elements, called gates, and basic memory elements, called flip-flops.
- The simplicity in digital circuit design is due to the fact that input and output signals of each gate or flip-flop can assume only two values, 0 and 1.
- The changes in signal values are governed by laws of Boolean algebra.
- The fact that Boolean algebra is finite and richer in properties than ordinary algebra leads to simple optimisation techniques for functions.
- In order to learn techniques for design of digital circuits, we must understand the properties of Boolean algebra.

# 3. Boolean Algebra
*- Binary Signals (1) -*

- Digital logic hides the analog world by mapping the infinite set of real values into 2 subsets (0 and 1).
- A logic value, 0 or 1, is often called a <u>binary digit</u> (<u>bit</u>).
- With n bits, $2^n$ different entities are represented.
- When using electronic circuits, digital designers often use the words “LOW” and “HIGH”, in place of “0” and “1”.
- The assignment of 0 to LOW and 1 to HIGH is called <u>positive logic</u>. The opposite assignment is called <u>negative logic</u>.
- Other technologies can be used to represent bits with physical states.

# 3. Boolean Algebra
*- Binary Signals (2) -*

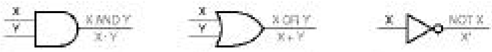
| Technology | State Representing Bit | |
|---|---|---|
| | 0 | 1 |
| Pneumatic logic | Fluid at low pressure | Fluid at high pressure |
| Relay logic | Circuit open | Circuit closed |
| Complementary metal-oxide semiconductor (CMOS) logic | 0–1.5 V | 3.5–5.0 V |
| Transistor-transistor logic (TTL) | 0–0.8 V | 2.0–5.0 V |
| Fiber optics | Light off | Light on |
| Dynamic memory | Capacitor discharged | Capacitor charged |
| Nonvolatile, erasable memory | Electrons trapped | Electrons released |
| Bipolar read-only memory | Fuse blown | Fuse intact |
| Bubble memory | No magnetic bubble | Bubble present |
| Magnetic tape or disk | Flux direction “north” | Flux direction “south” |
| Polymer memory | Molecule in state A | Molecule in state B |
| Read-only compact disc | No pit | Pit |
| Rewriteable compact disc | Dye in crystalline state | Dye in noncrystalline state |

# 3. Boolean Algebra
*- Combinational vs. Sequential Systems -*

- A <u>combinational</u> logic system is one whose outputs depend only on its current inputs.
- A combinational system can be described by a <u>truth table</u>.
- The outputs of a <u>sequential</u> logic circuit depend not only on the current inputs but also on the past sequence of inputs $\Rightarrow$ memory.
- A sequential system can be described by a <u>state table</u>.
- A combinational system may contain any number of logic gates but no feedback loops.
- A <u>feedback loop</u> is a signal path of a circuit that allows the output of a gate to propagate back to the input of that same gate.
- Feedback loops generally create sequential circuit behaviour.

# 3. Boolean Algebra
*- Gates (1) -*

- Three basic gates (AND, OR, NOT) are sufficient to build any combinational digital logic system. They form a complete set.




| X | Y | X AND Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

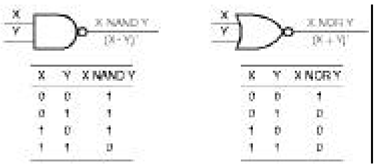
| X | Y | X OR Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| X | NOT X |
|---|---|
| 0 | 1 |
| 1 | 0 |

- The symbols and truth tables for AND and OR may be extended to gates with any number of inputs.
- The <u>bubble</u> on the inverter output denotes “inverting” behaviour.

# 3. Boolean Algebra
*- Gates (2) -*

- Two more logic functions are obtained by combining NOT with an AND or OR function in a single gate.



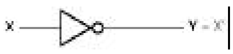

- The symbols and truth tables for NAND and NOR may also be extended to gates with any number of inputs.

# 3. Boolean Algebra
*- Switching Algebra -*

- In 1854, G. Boole (1815-1865) introduced the formalism that we use for the systematic treatment of logic which is now called <u>Boolean Algebra</u>.
- In 1938, C. Shannon (1916-2001) applied this algebra to prove that the properties of electrical switching circuits can be represented by a 2-valued Boolean Algebra, which is called <u>Switching Algebra.</u>
- Using this algebra, one can formulate propositions that are true or false, combine them to make new propositions and determine if the new propositions are true or false.
- We use a symbolic variable (ex. X) to represent the condition of a logic signal, which is in one of two possible values ( “0” or “1”).

# 3. Boolean Algebra
*- Axioms (1) -*

- The <u>axioms</u> (or postulates) of a mathematical system are a minimal set of basic definitions that we assume to be true.
- The first axioms embody the digital abstraction:
  (A1) X=0 if X≠1 (A1’) X=1 if X≠0
- We stated these axioms as a pair, the only difference being the interchange of the symbols 0 and 1.
- This applies to all the axioms and is the basis of <u>duality</u>.
- The next axioms embody the complement notation:
  (A2) If X=0, then X’=1 (A2’) If X=1, then X’=0




- We use a prime (’) to denote an inverter’s function.

# 3. Boolean Algebra
*- Axioms (2) -*

- The last three pairs of axioms state the formal definitions of the AND (logical multiplication) and OR (logical addition) operations:
  (A3) 0·0 = 0 (A3’) 1+1 = 1
  (A4) 1·1 = 1 (A4’) 0+0 = 0
  (A5) 0·1 = 1·0 = 0 (A5’) 1+0 = 0+1 = 1


- By convention, in a logic expression involving both multiplication and addition, multiplication has precedence.
- The expression X·Y+Y·Z’ is equivalent to (X·Y) + (Y·Z’).
- The axioms (A1-A5, A1’-A5’) completely define Boolean algebra.

# 3. Boolean Algebra
*- Theorems (1) -*

- <u>Theorems</u> are statements, known to be true, that allow us to manipulate algebraic expressions to have simpler analysis or more efficient synthesis of the corresponding circuits.
- Theorems involving a single variable:

| | | |
|---|---|---|
| (T1) X+0 = X | (T1’) X·1 = X | (Identities) |
| (T2) X+1 = 1 | (T2’) X·0 = 0 | (Null elements) |
| (T3) X+X = X | (T3’) X·X = X | (Idempotency) |
| (T4) (X’)’ = X | | (Involution) |
| (T5) X+X’ = 1 | (T5’) X·X’ = 0 | (Complements) |

- These theorems can be proved to be true. Let us prove T1:
  [X=0] 0+0=0 (true, according to A4’)
  [X=1] 1+0=1 (true, according to A5’)

# 3. Boolean Algebra
*- Theorems (2) -*

- Theorems involving two or three variables:

| | | |
|---|---|---|
| (T6) X+Y = Y+X | (T6’) X·Y = Y·X | (Commutativity) |
| (T7) (X+Y)+Z = X+(Y+Z) | (T7’) (X·Y)·Z = X·(Y·Z) | (Associativity) |
| (T8) X·Y+X·Z = X·(Y+Z) | (T8’) (X+Y)·(X+Z) = X+Y·Z | (Distributivity) |
| (T9) X+X·Y = X | (T9’) X·(X+Y) = X | (Covering) |
| (T10) X·Y+X·Y’ = X | (T10’) (X+Y)·(X+Y’) = X | (Combining) |
| (T11) X·Y+X’·Z+Y·Z = X·Y+X’·Z | | (Consensus) |
| (T11’) (X+Y)·(X’+Z)·(Y+Z) = (X+Y)·(X’+Z) | | |

- Attention to theorem T8’ which is not true for integers and reals.
- T9 and T10 are used in the minimisation of logic functions.

# 3. Boolean Algebra

*- Theorems (3) -*

- Several important theorems are true for an arbitrary number of variables.
- Theorems involving n variables:

| | |
|---|---|
| (T12) $X+X+ \ldots +X = X$<br>(T12') $X \cdot X \cdot \ldots \cdot X = X$ | Generalised Idempotency |
| (T13) $(X_1 \cdot X_2 \cdot \ldots \cdot X_n)' = X_1'+X_2'+ \ldots +X_n'$<br>(T13') $(X_1+X_2+ \ldots +X_n)' = X_1' \cdot X_2' \cdot \ldots \cdot X_n'$ | DeMorgan's theorems |
| (T14) $[F(X_1,X_2,\ldots,X_n,+,\cdot)]' = F(X_1',X_2',\ldots,X_n',\cdot,+)$ | Generalised DeMorgan's th. |
| (T15) $F(X_1,X_2,\ldots,X_n) = X_1 \cdot F(1,X_2,\ldots,X_n) + X_1' \cdot F(0,X_2,\ldots,X_n)$<br>(T15') $F(X_1,X_2,\ldots,X_n) = [X_1+F(0,X_2,\ldots,X_n)] \cdot [X_1'+F(1,X_2,\ldots,X_n)]$ | Shannon's expansion theorems |

# 3. Boolean Algebra

*- Theorems (4) -*

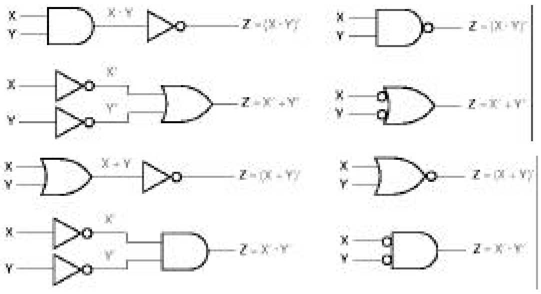
- DeMorgan's theorem (T13 and T13') for n=2:
  $(X \cdot Y)' = X'+Y'$
  $(X+Y)' = X' \cdot Y'$
- DeMorgan's theorem gives a procedure for complementing a logic function.
- DeMorgan's theorem can be used to convert AND/OR expressions to OR/AND expressions.
- Example:
  $Z = A'\,B'\,C \;+\; A'\,B\,C \;+\; A\,B'\,C \;+\; A\,B\,C'$
  $Z' = (A + B + C') \cdot (A + B' + C') \cdot (A' + B + C') \cdot (A' + B' + C)$

Augustus De Morgan (1806-1871)

# 3. Boolean Algebra

*- Theorems (5) -*

Equivalent gates according to DeMorgan's theorem



# 3. Boolean Algebra

*- Theorems (6) -*

- Since Boolean algebra has only two elements, we can also show the validity of these theorems by using truth tables.
- To do this, a truth table is built for each side of the equation that appears in the theorem.
- Then both sides of the equation are checked to see if they yield identical results for all the combinations of variable values.

| X | Y | $\overline{X}$ | $\overline{Y}$ | $\overline{X+Y}$ | $\overline{X}\bullet\overline{Y}$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |

- Let us prove DeMorgan's theorem (T13 and T13') for n=2:
  $(X+Y)' = X' \cdot Y'$ $\qquad$ $(X+Y)' = X' \cdot Y'$

| X | Y | $\overline{X}$ | $\overline{Y}$ | $\overline{X\bullet Y}$ | $\overline{X}+\overline{Y}$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |

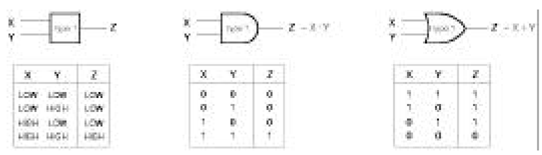
# 3. Boolean Algebra

*- Duality (1) -*

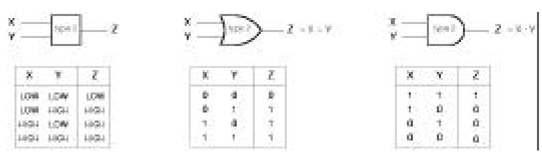
- Theorems were presented in pairs.
- The prime version of a theorem is obtained from the unprimed version by swapping "0" and "1", and "·" and "+".
- <u>Principle of Duality</u>: Any theorem or identity in Boolean algebra remains true if 0 and 1 are swapped and · and + are swapped throughout.
- Duality is important because it doubles the usefulness of everything about Boolean algebra and manipulation of logic functions.
- The <u>dual</u> of a logic expression is the same expression with + and · swapped: $F^D(X_1,X_2,\ldots,X_n,+,\cdot,') = F(X_1,X_2,\ldots,X_n,\cdot,+,')$.
- Do not confuse duality with DeMorgan's theorems!
  $[F(X_1,X_2,\ldots,X_n,+,\cdot)]' = F(X_1',X_2',\ldots,X_n',\cdot,+)$
  $[F(X_1,X_2,\ldots,X_n)]' = F^D(X_1',X_2',\ldots,X_n')$

# 3. Boolean Algebra

*- Duality (2) -*

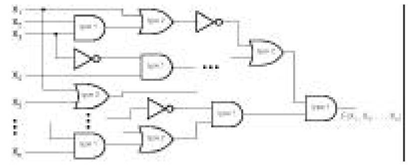


Electric function $\qquad$ Positive-logic $\qquad$ Negative-logic

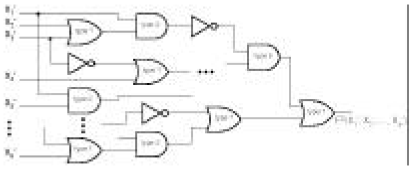


# 3. Boolean Algebra
*- Duality (3) -*

- Positive-logic



- Negative-logic



# 3. Boolean Algebra
*- Standard Representation (1) -*

- The most basic representation of a logic function is a truth table.
- A <u>truth table</u> lists the output of the circuit for every possible input combination.
- There are $2^n$ rows in a truth table for an n-variable function.

| Row | X | Y | Z | F |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | F(0,0,0) |
| 1 | 0 | 0 | 1 | F(0,0,1) |
| 2 | 0 | 1 | 0 | F(0,1,0) |
| 3 | 0 | 1 | 1 | F(0,1,1) |
| 4 | 1 | 0 | 0 | F(1,0,0) |
| 5 | 1 | 0 | 1 | F(1,0,1) |
| 6 | 1 | 1 | 0 | F(1,1,0) |
| 7 | 1 | 1 | 1 | F(1,1,1) |

| Row | X | Y | Z | F |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 1 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

- There are $2^8$ ($8=2^3$) different logic functions for 3 variables.

# 3. Boolean Algebra
*- Standard Representation (2) -*

- Truth tables can be converted to algebraic expressions.
- A <u>literal</u> is a variable or the complement of a variable. Ex: X, Y, X'.
- A <u>product term</u> is a single literal or a logical product of two or more literals. Ex: Z', W·X·Y, W·X'·Y'.
- A <u>sum-of-products</u> (SOP) is a logical sum of product terms. Ex: Z' + W·X·Y.
- A <u>sum term</u> is a single literal or a logical sum of two or more literals. Ex: Z', W+X+Y, W+X'+Y'.
- A <u>product-of-sums</u> (POS) is a logical product of sum terms. Ex: Z' · (W+X+Y).

# 3. Boolean Algebra
*- Standard Representation (3) -*

- A <u>normal term</u> is a product or sum term in which no variable appears more than once.
  Examples (**non**-normal terms): W·X·X'·Z', W'+Y'+Z+W'.
- A n-variable <u>minterm</u> is a normal product term with n literals.
  Examples (with 4 variables): W·X·Y·Z', W'·X'·Y·Z.
- A n-variable <u>maxterm</u> is a normal sum term with n literals.
  Examples (with 4 variables): W+X+Y+Z', W'+X'+Y+Z.
- There is a correspondence between the truth table and minterms and maxterms.
- A minterm is a product term that is 1 in one row of the truth table.
- A maxterm is a sum term that is 0 in one row of the truth table.

# 3. Boolean Algebra
*- Standard Representation (4) -*

Minterms and maxterms for a 3-variable function F(X,Y,Z)

| Row | X | Y | Z | F | Minterm | Maxterm |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | F(0,0,0) | X'·Y'·Z' | X+Y+Z |
| 1 | 0 | 0 | 1 | F(0,0,1) | X'·Y'·Z | X+Y+Z' |
| 2 | 0 | 1 | 0 | F(0,1,0) | X'·Y·Z' | X+Y'+Z |
| 3 | 0 | 1 | 1 | F(0,1,1) | X'·Y·Z | X+Y'+Z' |
| 4 | 1 | 0 | 0 | F(1,0,0) | X·Y'·Z' | X'+Y+Z |
| 5 | 1 | 0 | 1 | F(1,0,1) | X·Y'·Z | X'+Y+Z' |
| 6 | 1 | 1 | 0 | F(1,1,0) | X·Y·Z' | X'+Y'+Z |
| 7 | 1 | 1 | 1 | F(1,1,1) | X·Y·Z | X'+Y'+Z' |

# 3. Boolean Algebra
*- Standard Representation (5) -*

- An n-variable minterm can be represented by an n-bit integer (the <u>minterm number</u>).
- In minterm i, a variable appears complemented if the respective bit in the binary representation of i is 0; otherwise it is uncomplemented.
- For example, row 5 (101) is related to minterm X·Y'·Z.
- In maxterm i, a variable appears complemented if the corresponding bit in the binary representation of i is 1; otherwise it is unprimed.
- For example, row 5 (101) is related to maxterm X'+Y+Z'.
- To specify the minterms and maxterms, it is mandatory to know the number of variables in the function and their order.

# 3. Boolean Algebra

*- Standard Representation (6) -*

| Row | X | Y | Z | F |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 1 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

- Based on the correspondence between the truth table and the <u>minterms</u>, an algebraic representation of a logic function can be created.
- The <u>canonical sum</u> of a logic function is a sum of the minterms corresponding to truth table rows for which the function is 1.
- From the table:
  $F = \sum_{X,Y,Z} (0,3,4,6,7) = X'\cdot Y'\cdot Z' + X'\cdot Y\cdot Z + X\cdot Y'\cdot Z' + X\cdot Y\cdot Z' + X\cdot Y\cdot Z$
- The notation $\sum_{X,Y,Z} (0,3,4,6,7)$ is a <u>minterm list</u> and means the sum of minterms 0,3,4,6, and 7, with variables X, Y, and Z.
- The minterm list is also known as the <u>on-set</u> for the logic function.

# 3. Boolean Algebra

*- Standard Representation (7) -*

| Row | X | Y | Z | F |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 1 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

- Based on the correspondence between the truth table and the <u>maxterms</u>, an algebraic representation of a logic function can be created.
- The <u>canonical product</u> of a function is a product of the maxterms corresponding to input combinations for which the function is 0.
- From the table:
  $F = \prod_{X,Y,Z} (1,2,5) = (X+Y+Z') \cdot (X+Y'+Z) \cdot (X'+Y+Z')$
- The notation $\prod_{X,Y,Z} (1,2,5)$ is a <u>maxterm list</u> and means the product of maxterms 1,2, and 5, with variables X, Y, and Z.
- The maxterm list is also known as the <u>off-set</u> for the logic function.

# 3. Boolean Algebra

*- Standard Representation (8) -*

- It is easy to convert between a minterm list and a maxterm list.
- For a function of n variables, the minterms and maxterms are in the set $\{0, 1, \ldots, 2^n-1\}$.
- A minterm or maxterm list contains a subset of these numbers.
- To switch between the lists, one takes the set complement.
- Examples:
  $\sum_{A,B,C} (0,1,2,3) = \prod_{A,B,C} (4,5,6,7)$
  $\sum_{X,Y} (1) = \prod_{X,Y} (0,2,3)$
  $\sum_{W,X,Y,Z} (1,2,3,5,8,12,13) = \prod_{W,X,Y,Z} (0,4,6,7,9,10,11,14,15)$

# 3. Boolean Algebra

*- Standard Representation (9) -*

- We have learned 5 possible representations for a combinational logic function.
  - **A truth table;**
  - **An algebraic sum of minterms (the canonical sum);**
  - **A minterm list, using the $\sum$ notation;**
  - **An algebraic product of maxterms (the canonical product);**
  - **A maxterm list, using the $\prod$ notation;**
- Each one of these representations specifies exactly the same information.
- Given any of them, we can derive the other four using a simple mechanical process.

# 3. Boolean Algebra

*- Examples (1) -*

- Ex.1: Let $F = X\cdot Y + X\cdot Y'\cdot Z + X'\cdot Y\cdot Z$. Derive the expression for F' in the product of sums form.
- **F′ = (XY + XY′Z + X′YZ)′**
  **= (XY)′·(XY′Z)′·(X′YZ)′**
  **= (X′+Y′)(X′+Y+Z′)(X+Y′+Z′)**
- Ex.2: Express the function $G(X,Y,Z) = X + Y'\cdot Z$ as a sum of minterms.
- **G = X + Y·Z**
  **= X·(Y+Y′)·(Z+Z′) + Y·Z·(X+X′)**
  **= XYZ + XYZ′ + XY′Z + XY′Z′ + XYZ + X′YZ**
  **= X′YZ + XY′Z′ + XY′Z + XYZ′+ X′YZ**
  **=** $\sum_{X,Y,Z} (3,4,5,6,7)$

# 3. Boolean Algebra

*- Examples (2) -*

- Ex.3: Derive the product-of-maxterms form for $H = X'\cdot Y' + X\cdot Z$.
- **H = X′Y′ + XZ**
  **= (X′Y′+X)(X′Y′+Z)**
  **= (X′+X)(Y′+X)(X′+Z)(Y′+Z)**
  **= (X+Y′)(X′+Z)(Y′+Z)**
- Each OR term in the expression is missing one variable:
  - **X+Y′ = X+Y′+ZZ′ = (X+Y′+Z)(X+Y′+Z′)**
  - **X′+Z = X′+Z+YY′ = (X′+Y+Z)(X′+Y′+Z)**
  - **Y′+Z = Y′+Z+XX′ = (X+Y′+Z)(X′+Y′+Z)**
- Finally we combine these terms:
  **H = (X+Y′+Z)(X+Y′+Z′)(X′+Y+Z)(X′+Y′+Z)**
  $\prod_{X,Y,Z} (2,3,4,6)$

# 3. Boolean Algebra

*- Examples (3) -*

- Ex.4: Derive the product-of-maxterms form for H = X'·Y' + X·Z.

| X | Y | Z | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

- From the table, we obtain:
  **H =** $\prod_{X,Y,Z}$ (2,3,4,6)
  **H =** $\sum_{X,Y,Z}$ (0,1,5,7)
- Compare this solution with the solution of ex.3.

- Ex.5: Derive a standard form with a reduced number of operators for J = XYZ + XYZ' + XY'Z + X'YZ.
- **J = XYZ + XYZ' + XYZ + XY'Z + XYZ + X'YZ**
  **XY(Z+Z') + X(Y+Y')Z + (X+X')YZ**
  **XY'+XZ+YZ**